

INSTANTANEOUS EXPONENTIAL LOWER BOUND FOR SOLUTIONS TO THE BOLTZMANN EQUATION WITH MAXWELLIAN DIFFUSION BOUNDARY CONDITIONS

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ABSTRACT. We prove the immediate appearance of an exponential lower bound, uniform in time and space, for continuous mild solutions to the full Boltzmann equation in a C^2 convex bounded domain with the physical Maxwellian diffusion boundary conditions, under the sole assumption of regularity of the solution. We investigate a wide range of collision kernels, with and without Grad's angular cutoff assumption. In particular, the lower bound is proven to be Maxwellian in the case of cutoff collision kernels. Moreover, these results are entirely constructive if the initial distribution contains no vacuum, with explicit constants depending only on the *a priori* bounds on the solution.

Keywords: Boltzmann equation, Exponential lower bound, Maxwellian lower bound, Maxwellian diffusion boundary conditions.

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1. INTRODUCTION

The Boltzmann equation rules the dynamics of rarefied gas particles moving in a domain Ω of \mathbb{R}^d with velocities in \mathbb{R}^d ($d \geq 2$) when the only interactions taken into account are elastic binary collisions. More precisely, the Boltzmann equation describes the time evolution of $f(t, x, v)$, the distribution of particles in position and velocity, starting from an initial distribution $f_0(x, v)$.

In the present article we are interested in the case where the gas stays inside a domain of which walls are heated at a constant temperature T_∂ . Contrary to the classical specular (billiard balls) or bounce-back reflections boundary conditions, the temperature of the boundary generates a diffusion towards the inside of the domain which prevents the usual preservation of energy of the gas.

We investigate the case where Ω is a C^2 convex bounded domain and that the boundary conditions are Maxwellian diffusion. The Boltzmann equation reads

$$(1.1) \quad \begin{aligned} \forall t \geq 0 \quad , \quad \forall (x, v) \in \Omega \times \mathbb{R}^d, \quad \partial_t f + v \cdot \nabla_x f &= Q(f, f), \\ \forall (x, v) \in \overline{\Omega} \times \mathbb{R}^d, \quad f(0, x, v) &= f_0(x, v), \end{aligned}$$

with f satisfying the Maxwellian diffusion boundary condition:

$$\forall (t, x, v) \in \mathbb{R}^{*+} \times \partial\Omega \times \mathbb{R}^d, \quad f(t, x, v) = f_\partial(t, x, v),$$

where

$$(1.2) \quad f_\partial(t, x, v) = \left[\int_{v \cdot n(x) > 0} f(t, x, v) (v \cdot n(x)) \, dv \right] \frac{1}{(2\pi)^{\frac{d-1}{2}} T_\partial^{\frac{d+1}{2}}} e^{-\frac{|v|^2}{2T_\partial}},$$

with $n(x)$ denoting the outwards normal to $\overline{\Omega}$ at x on $\partial\Omega$. This boundary condition expresses the physical process where particles are absorbed by the wall and then emitted back into Ω according to the thermodynamical equilibrium distribution between the wall and the gas.

The operator $Q(f, f)$ encodes the physical properties of the interactions between two particles. This operator is quadratic and local in time and space. It is given by

$$Q(f, f) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B(|v - v_*|, \cos \theta) [f' f'_* - f f_*] \, dv_* d\sigma,$$

where f' , f_* , f'_* and f are the values taken by f at v' , v_* , v'_* and v respectively. Define:

$$\begin{cases} v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma \\ v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma \end{cases}, \text{ and } \cos \theta = \left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle.$$

We recognise here the conservation of kinetic energy and momentum when two particles of velocities v and v_* collide to give two particles of velocities v' and v'_* . The collision kernel $B \geq 0$ contains all the information about the interaction between

two particles and is determined by physics (see [3] or [4] for a formal derivation for the hard sphere model of particles). In this paper we shall only be interested in the case of B satisfying the following product form

$$(1.3) \quad B(|v - v_*|, \cos \theta) = \Phi(|v - v_*|) b(\cos \theta),$$

which is a common assumption as it is more convenient and also covers a wide range of physical applications. Moreover, we shall assume that Φ satisfies either

$$(1.4) \quad \forall z \in \mathbb{R}, \quad c_\Phi |z|^\gamma \leq \Phi(z) \leq C_\Phi |z|^\gamma$$

or a mollified assumption

$$(1.5) \quad \begin{cases} \forall |z| \geq 1 \in \mathbb{R}, & c_\Phi |z|^\gamma \leq \Phi(z) \leq C_\Phi |z|^\gamma \\ \forall |z| \leq 1 \in \mathbb{R}, & c_\Phi \leq \Phi(z) \leq C_\Phi, \end{cases}$$

c_Φ and C_Φ being strictly positive constants and γ in $(-d, 1]$. The collision kernel is said to be “hard potential” in the case of $\gamma > 0$, “soft potential” if $\gamma < 0$ and “Maxwellian” if $\gamma = 0$.

Finally, we shall consider b to be a continuous function on θ in $(0, \pi]$, strictly positive near $\theta \sim \pi/2$, which satisfies

$$(1.6) \quad b(\cos \theta) \sin^{d-2} \theta \underset{\theta \rightarrow 0^+}{\sim} b_0 \theta^{-(1+\nu)}$$

for $b_0 > 0$ and ν in $(-\infty, 2)$. The case when b is locally integrable, $\nu < 0$, is referred to by the Grad's cutoff assumption (first introduced in [7]) and therefore B will be said to be a cutoff collision kernel. The case $\nu \geq 0$ will be designated by non-cutoff collision kernel.

1.1. Motivations and comparison with previous results. The aim of this article is to show and to quantify the strict positivity of the solutions to the Boltzmann equation when the gas particles move in a domain with boundary conditions. In that sense, it continues the study started in [1] about exponential lower bounds for solutions to the Boltzmann equation when specular reflections boundary conditions were taken into account.

More precisely, we shall prove that continuous solutions to the Boltzmann equation, with Maxwellian diffusion boundary conditions in a C^2 convex bounded domain, which have uniformly bounded energy satisfy an immediate exponential lower bound:

$$\forall t \geq t_0, \forall (x, v) \in \mathbb{T}^d \times \mathbb{R}^d, \quad f(t, v) \geq C_1 e^{-C_2 |v|^K},$$

for all $t_0 > 0$. Moreover, in the case of collision kernel with angular cutoff we recover a Maxwellian lower bound

$$\forall \tau > 0, \exists \rho, \theta > 0, \forall t \geq \tau, \forall (x, v) \in \Omega \times \mathbb{R}^d, \quad f(t, x, v) \geq \frac{\rho}{(2\pi\theta)^{d/2}} e^{-\frac{|v|^2}{2\theta}}.$$

We would like to emphasize that, in the spirit of [1], our results show that the gas will instantaneously fill up the whole domain even if the initial configuration contains vacuum. Indeed, they only require some regularity on the solution and no further assumption on its local density. Previous studies assumed the latter to be uniformly bounded from below, which is equivalent of assuming *a priori* either that there is no vacuum or that the solution is strictly positive.

Moreover, the present results only require solutions to the Boltzmann equation to be continuous away from the grazing set

$$(1.7) \quad \Lambda_0 = \{(x, v) \in \partial\Omega \times \mathbb{R}^d, \quad n(x) \cdot v = 0\},$$

which is a property that is known to hold in the case of Maxwellian diffusion boundary conditions [9].

The issue of quantifying the positivity of solutions has been investigated for a long time since it not only presents a great physical interest but also appears to be of significant importance for the mathematical study of the Boltzmann equation. Indeed, exponential lower bounds are essential for entropy-entropy production methods used to describe long-time behaviour for kinetic equations [5][6]. More recently, such lower bounds were needed to prove uniqueness of solutions to the Boltzmann equation in $L_v^1 L_x^\infty (1 + |v|^{2+0})$ [8].

Several works quantified the study of an explicit lower bound for solutions to the Boltzmann equation. We give here a brief overview and we refer the interested reader to the more detailed description in [1].

The first result about the strict positivity of solutions to the Boltzmann equation was derived by Carleman [2]. Noticing that a part Q^+ of the Boltzmann operator Q satisfies a spreading property, roughly speaking

$$Q^+(\mathbf{1}_{B(\bar{v},r)}, \mathbf{1}_{B(\bar{v},r)}) \geq C_+ \mathbf{1}_{B(\bar{v},\sqrt{2}r)},$$

with $C_+ < 1$ (see Lemma 3.2 for an exact statement), he proved the immediate creation of an exponential lower bound for a certain class of solutions (radially symmetric in velocity) to the spatially homogeneous equation with hard potential kernel with angular cutoff. The latter result was improved to a Maxwellian lower bound and extended to the case of non-radially symmetric solutions to the spatially homogeneous equation with hard potential and cutoff by Pulvirenti and Wennberg [11].

Finally, the study in the case of the full equation has been tackled by Mouhot [10] in the case of the torus $\Omega = \mathbb{T}^d$, and more recently by the author [1] in C^2 convex bounded domains with specular boundary conditions in all dimension d . In both articles, a Maxwellian lower bound is derived for solutions to the full Boltzmann equation with angular cutoff (both with hard or soft potentials) and they showed the appearance of an exponential lower bound in the non-cutoff case.

Our present results show that the previous properties proven for the full Boltzmann equation [10][1] still hold in the physically relevant case of the Maxwellian diffusion generated by the boundary of the domain Ω . This is done for all the physically relevant collision kernels such as with and without angular cutoff and hard and soft potentials. Moreover, in the case of a solution that has uniformly bounded local mass and entropy, the proofs are entirely constructive and the constants are explicit and only depend on the *a priori* bounds on the solution and the geometry of the domain, which is of great physical interest for the study of the spreading of gas into a domain with heated walls.

There are two key contributions in the present article. The main one is a quantification of the strict positivity of the Maxwellian diffusion process thanks to a combination of a localised positivity of the solution and a geometrical study of the rebounds against a convex boundary. Roughly speaking, we show that the wall instantaneously starts to diffuse in all directions and that its diffusion is uniformly bounded from below. The second one is a spreading method combining the effects used in previous studies and the exponential diffusive process (see next section for details).

1.2. Our strategy. The main strategy to tackle this result relies on the breakthrough of Carleman [2], namely finding an “upheaval point” (a first minoration uniform in time but localised in velocity) and spreading this bound, thanks to the spreading property of the Q^+ operator, in order to include larger and larger velocities and finally compare it to an exponential. The case of the spatially non-homogeneous equation [10][1] was dealt by finding a spreading method that was invariant along the flow of characteristics.

The creation of “upheaval points” (localised in space because of boundary effects) is essentially the method developed in [1] for general continuous initial datum or the

one of [11][10] for constructive purposes. There is a new technical treatment of the time of appearance of such lower bounds but does not present any new difficulties.

The main issue is the use of a spreading method that would be invariant along the flow of characteristics [10][1]. In the case of Maxwellian diffusion boundary conditions characteristic trajectories are no longer well defined. Indeed, once a trajectory touches the boundary it is absorbed and re-emitted in all the directions. Characteristic trajectories are therefore only defined in between two consecutive rebounds and one cannot hope to use the invariant arguments developed in [10][1].

The case of the torus, studied in [10], indicates that without boundaries the exponential lower bound is created after time $t = 0$ as quickly as one wants. In the case of a bounded domain with specular reflection boundary conditions [1], this minoration also occurs immediately. Together, it roughly means that one can expect to obtain an exponential lower bound on straight lines uniformly on how close the particle is from the boundary. Therefore we expect the same kind of uniform bounds to arise on each characteristic trajectory in between two consecutive rebounds as long as the Maxwellian diffusion emitted by the boundary is uniformly non-negative.

Our strategy is therefore to first prove that the boundary condition produces a strictly positive quantity uniformly towards the interior of Ω and then to find a method to spread either this diffusion or the localised “upheaval points”. More precisely, if there is no contact during a time $\tau > 0$ we expect to be able to use the spreading method developed in [10] from the initial lower bounds. Else there is a contact during the interval $[0, \tau]$ we cannot hope to use the latter spreading method, nor its more general characteristics invariant version derived in [1], since the Maxwellian diffusion boundary condition acts like an absorption for particles arriving on the boundary. But this boundary condition also diffuses towards the interior of the domain Ω what it has absorbed. This diffusion follows an exponential law and therefore of the shape of a Maxwellian lower bound that we manage to keep along the incoming characteristic trajectory.

Collision kernels satisfying a cutoff property as well as collision kernels with a non-cutoff property will be treated following the strategy described above. The only difference is the decomposition of the Boltzmann bilinear operator Q we consider in each case. In the case of a non-cutoff collision kernel, we shall divide it into a cutoff collision kernel and a remainder. The cutoff part will already be dealt with and a careful control of the L^∞ -norm of the remainder will give us the expected exponential lower bound, which decreases faster than a Maxwellian.

1.3. Organisation of the paper. Section 2 is dedicated to the statement and the description of the main results proven in this article. It contains three different parts

Section 2.1 defines all the notations which will be used throughout the article.

The last subsections, 2.2 and 2.3, are dedicated to a mathematical formulation of the results related to the lower bound in, respectively, the cutoff case and the non-cutoff case, described above. It also defines the concept of mild solutions to the Boltzmann equation in each case.

Section 3 deals with the case of the immediate Maxwellian lower bound for collision kernels with angular cutoff. As described in our strategy, it follows three steps.

Section 3.1 generates the localised “upheaval points” for general initial datum. A constructive approach to that problem is given in Section 3.4.

The uniform positivity of the Maxwellian diffusion is proven in Section 3.2.

Finally, Section 3.3 combines the standard spreading methods with the diffusion process to prove the expected instantaneous Maxwellian lower bound.

To conclude, Section 4 proves the immediate appearance of an exponential lower bound in the case of collision kernels without angular cutoff. We emphasize here that this Section actually explains the adaptations required compared to the case of collision kernels with the cutoff property.

2. MAIN RESULTS

We begin with the notations we shall use all along this article.

2.1. Notations. First of all, we denote $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$ and $y^+ = \max\{0, y\}$, the positive part of y .

Functional spaces. This study will hold in specific functional spaces regarding the v variable that we describe here and use throughout the sequel. For all p in $[1, \infty]$, we use the shorthand notation for Lebesgue spaces $L_v^p = L^p(\mathbb{R}^d)$.

For $p \in [1, \infty]$ and $k \in \mathbb{N}$ we use the Sobolev spaces $W_v^{k,p}$ by the norm

$$\|f\|_{W_v^{k,p}} = \left[\sum_{|s| \leq k} \|\partial^s f(v)\|_{L_v^p}^p \right]^{1/p}.$$

Physical observables and hypotheses. In the sequel of this study, we are going to need bounds on some physical observables of solution to the Boltzmann equation (1.1).

We consider here a function $f(t, x, v) \geq 0$ defined on $[0, T) \times \Omega \times \mathbb{R}^d$ and we recall the definitions of its local hydrodynamical quantities.

- its local energy

$$e_f(t, x) = \int_{\mathbb{R}^d} |v|^2 f(t, x, v) dv,$$

- its local weighted energy

$$e'_f(t, x) = \int_{\mathbb{R}^d} |v|^{\tilde{\gamma}} f(t, x, v) dv,$$

where $\tilde{\gamma} = (2 + \gamma)^+$,

- its local L^p norm ($p \in [1, +\infty)$)

$$l_f^p(t, x) = \|f(t, x, \cdot)\|_{L_v^p},$$

- its local $W^{2,\infty}$ norm

$$w_f(t, x) = \|f(t, x, \cdot)\|_{W_v^{2,\infty}}.$$

Our results depend on uniform bounds on those quantities and therefore, to shorten calculations we will use the following

$$\begin{aligned} E_f &= \sup_{(t,x) \in [0,T) \times \Omega} e_f(t, x) \quad , \quad E'_f = \sup_{(t,x) \in [0,T) \times \Omega} e'_f(t, x), \\ L_f^p &= \sup_{(t,x) \in [0,T) \times \Omega} l_f^p(t, x) \quad , \quad W_f = \sup_{(t,x) \in [0,T) \times \Omega} w_f(t, x). \end{aligned}$$

In our theorems we are giving a priori lower bound results for solutions to (1.1) satisfying some properties about their local hydrodynamical quantities. Those properties will differ depending on which case of collision kernel we are considering. We will take them as assumptions in our proofs and they are the following.

- In the case of hard or Maxwellian potentials with cutoff ($\gamma \geq 0$ and $\nu < 0$):

$$(2.1) \quad E_f < +\infty.$$

- In the case of a singularity of the kinetic collision kernel ($\gamma \in (-d, 0)$) we shall make the additional assumption

$$(2.2) \quad L_f^{p_\gamma} < +\infty,$$

where $p_\gamma > d/(d + \gamma)$.

- In the case of a singularity of the angular collision kernel ($\nu \in [0, 2)$) we shall make the additional assumption

$$(2.3) \quad W_f < +\infty, \quad E'_f < +\infty.$$

Assumption (2.2) implies the boundedness of the local entropy and if $\gamma \leq 0$ we have $E'_f \leq E_f$ and so in some cases several assumptions might be redundant.

Moreover, in the case of the torus with periodic conditions or the case of bounded domain with specular boundary reflections [1], solutions to (1.1) also satisfied the conservation of the total mass and the total energy. The case with Maxwellian diffusion boundary conditions only preserves, in general (see [12] for instance), the total mass:

$$(2.4) \quad \exists M \geq 0, \quad \forall t \in \mathbb{R}^+, \quad \int_{\Omega} \int_{\mathbb{R}^d} f(t, x, v) dx dv = M.$$

Characteristic trajectories. The characteristic trajectories of the equation are only defined between two consecutive rebounds against the boundary and they are given by straight lines that we will denote by

$$(2.5) \quad \forall 0 \leq s \leq t, \quad \forall (x, v) \in \mathbb{R}^d \times \mathbb{R}^d, \quad X_{s,t}(x, v) = x - (t - s)v.$$

Because $\overline{\Omega}$ is a closed set, we can define the first time of contact between a backward trajectory and $\partial\Omega$:

$$(2.6) \quad \forall (x, v) \in \overline{\Omega} \times \mathbb{R}^d, \quad t_{\partial}(x, v) = \min \{t \geq 0 : x - vt \in \partial\Omega\},$$

and the contact point between such a trajectory and the boundary:

$$(2.7) \quad \forall (x, v) \in \overline{\Omega} \times \mathbb{R}^d, \quad x_{\partial}(x, v) = x - vt_{\partial}(x, v)$$

2.2. Maxwellian lower bound for cutoff collision kernels. The final theorem we prove in the case of cutoff collision kernel is the immediate appearance of a uniform Maxwellian lower bound. We use, in that case, the Grad's splitting for the bilinear operator Q such that the Boltzmann equation reads

$$\begin{aligned} Q(g, h) &= \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \Phi(|v - v_*|) b(\cos \theta) [h' g'_* - h g_*] dv_* d\sigma \\ &= Q^+(g, h) - Q^-(g, h), \end{aligned}$$

where we used the following definitions

$$\begin{aligned} Q^+(g, h) &= \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \Phi(|v - v_*|) b(\cos \theta) h' g'_* dv_* d\sigma, \\ (2.8) \quad Q^-(g, h) &= n_b (\Phi * g(v)) h = L[g](v) h, \end{aligned}$$

where

$$(2.9) \quad n_b = \int_{\mathbb{S}^{d-1}} b(\cos \theta) d\sigma = |\mathbb{S}^{d-2}| \int_0^\pi b(\cos \theta) \sin^{d-2} \theta d\theta.$$

As already mentionned, the characteristics of our problem can only be defined in between two consecutive rebounds against $\partial\Omega$. We can therefore define a mild solution of the Boltzmann equation in the cutoff case, which is expressed by a Duhamel formula along the characteristics. This weaker form of solutions is actually the key point for our result and also gives a more general statement.

Definition 2.1. Let f_0 be a measurable function non-negative almost everywhere on $\overline{\Omega} \times \mathbb{R}^d$.

A measurable function $f = f(t, x, v)$ on $[0, T) \times \overline{\Omega} \times \mathbb{R}^d$ is a mild solution of the Boltzmann equation associated to the initial datum $f_0(x, v)$ if

- (1) f is non-negative on $\overline{\Omega} \times \mathbb{R}^d$,
- (2) for every (t, x, v) in $\mathbb{R}^+ \times \Omega \times \mathbb{R}^d$:

$$s \mapsto L[f(t, X_{s,t}(x, v), \cdot)](v), \quad t \mapsto Q^+[f(t, X_{s,t}(x, v), \cdot), f(t, X_{s,t}(x, v), \cdot)](v)$$

are in $L^1_{loc}([0, T))$,

- (3) and for each $t \in [0, T)$, for all $x \in \Omega$ and $v \in \mathbb{R}^d$

$$\begin{aligned} (2.10) \quad f(t, x, v) &= f_0(x - vt, v) \exp \left[- \int_0^t L[f(s, X_{s,t}(x, v), \cdot)](v) ds \right] \\ &\quad + \int_0^t \exp \left(- \int_s^t L[f(s', X_{s',s}(x, v), \cdot)](v) ds' \right) \\ &\quad \times Q^+[f(s, X_{s,t}(x, v), \cdot), f(s, X_{s,t}(x, v), \cdot)](v) ds. \end{aligned}$$

if $t \leq t_\partial(x, v)$ or else

$$\begin{aligned} (2.11) \quad f(t, x, v) &= f_\partial(t_\partial(x, v), x_\partial(x, v), v) \exp \left[- \int_{t_\partial(x, v)}^t L[f(s, X_{s,t}(x, v), \cdot)](v) ds \right] \\ &\quad + \int_{t_\partial(x, v)}^t \exp \left(- \int_s^t L[f(s', X_{s',s}(x, v), \cdot)](v) ds' \right) \\ &\quad \times Q^+[f(s, X_{s,t}(x, v), \cdot), f(s, X_{s,t}(x, v), \cdot)](v) ds. \end{aligned}$$

Now we state our result.

Theorem 2.2. *Let Ω be a C^2 open bounded domain in \mathbb{R}^d with nowhere null normal vector and let f_0 be a non-negative continuous function on $\overline{\Omega} \times \mathbb{R}^d$. Let $B = \Phi b$ be a collision kernel satisfying (1.3), with Φ satisfying (1.4) or (1.5) and b satisfying (1.6) with $\nu < 0$. Let $f(t, x, v)$ be a mild solution of the Boltzmann equation in $\overline{\Omega} \times \mathbb{R}^d$ on some time intervalle $[0, T)$, $T \in (0, +\infty]$, which satisfies*

- *f is continuous on $[0, T) \times (\overline{\Omega} \times \mathbb{R}^d - \Lambda_0)$ (Λ_0 grazing set defined by (1.7)), $f(0, x, v) = f_0(x, v)$ and $M > 0$ in (2.4);*
- *if Φ satisfies (1.4) with $\gamma \geq 0$ or if Φ satisfies (1.5), then f satisfies (2.1);*
- *if Φ satisfies (1.4) with $\gamma < 0$, then f satisfies (2.1) and (2.2).*

Then for all $\tau \in (0, T)$ there exists $\rho > 0$ and $\theta > 0$, depending on τ , E_f (and $L_f^{p\gamma}$ if Φ satisfies (1.4) with $\gamma < 0$), such that for all $t \in [\tau, T)$ the solution f is bounded from below, almost everywhere, by a global Maxwellian distribution with density ρ and temperature θ , i.e.

$$\forall t \in [\tau, T), \forall (x, v) \in \overline{\Omega} \times \mathbb{R}^d, \quad f(t, x, v) \geq \frac{\rho}{(2\pi\theta)^{d/2}} e^{-\frac{|v|^2}{2\theta}}.$$

If we add the assumptions of uniform boundedness of f_0 and of the local mass and entropy of the solution f we can use the arguments originated in [11] to construct explicitly the initial “upheaval point”, without any compactness argument. We refer the reader to Section 3.4 which gives the following corollary.

Corollary 2.3. *Suppose that conditions of Theorem 2.2 are satisfied and further assume that f_0 is uniformly bounded from below*

$$\forall (x, v) \in \Omega \times \mathbb{R}^d, \quad f_0(x, v) \geq \varphi(v) > 0,$$

and that f has a bounded local mass and entropy

$$\begin{aligned} R_f &= \inf_{(t,x) \in [0,T) \times \Omega} \int_{\mathbb{R}^d} f(t, x, v) dv > 0 \\ H_f &= \sup_{(t,x) \in [0,T) \times \Omega} \left| \int_{\mathbb{R}^d} f(t, x, v) \log f(t, x, v) dv \right| < +\infty. \end{aligned}$$

Then conclusion of Theorem 2.2 holds true with the constants ρ and θ being explicitly constructed in terms of τ , E_f , H_f and $L_f^{p\gamma}$.

2.3. Exponential lower bound for non-cutoff collision kernels. In the case of non-cutoff collision kernels ($0 \leq \nu < 2$ in (1.6)), Grad’s splitting does not make sense anymore and so we have to find a new way to define mild solutions to the Boltzmann equation (1.1). The splitting we are going to use is a standard one and it reads

$$\begin{aligned} Q(g, h) &= \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \Phi(|v - v_*|) b(\cos\theta) [h'g'_* - hg_*] dv_* d\sigma \\ &= Q_b^1(g, h) - Q_b^2(g, h), \end{aligned}$$

where we used the following definitions

$$\begin{aligned}
 Q_b^1(g, h) &= \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \Phi(|v - v_*|) b(\cos\theta) g'_*(h' - h) dv_* d\sigma, \\
 (2.12) \quad Q_b^2(g, h) &= - \left(\int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \Phi(|v - v_*|) b(\cos\theta) [g'_* - g_*] dv_* d\sigma \right) h \\
 &= S[g](v)h.
 \end{aligned}$$

We would like to use the properties we derived in the study of collision kernels with cutoff. Therefore we will consider additional splitting of Q .

For ε in $(0, \pi/4)$ we define a cutoff angular collision kernel

$$b_\varepsilon^{CO}(\cos\theta) = b(\cos\theta) \mathbf{1}_{|\theta| \geq \varepsilon}$$

and a non-cutoff one

$$b_\varepsilon^{NCO}(\cos\theta) = b(\cos\theta) \mathbf{1}_{|\theta| \leq \varepsilon}.$$

Considering the two collision kernels $B_\varepsilon^{CO} = \Phi b_\varepsilon^{CO}$ and $B_\varepsilon^{NCO} = \Phi b_\varepsilon^{NCO}$, we can combine Grad's splitting (2.8) applied to B_ε^{CO} with the non-cutoff splitting (2.12) applied to B_ε^{NCO} . This yields the splitting we shall use to deal with non-cutoff collision kernels,

$$(2.13) \quad Q = Q_\varepsilon^+ - Q_\varepsilon^- + Q_\varepsilon^1 - Q_\varepsilon^2,$$

where we use the shortened notations $Q_\varepsilon^\pm = Q_{b_\varepsilon^{CO}}^\pm$ and $Q_\varepsilon^i = Q_{b_\varepsilon^{NCO}}^i$, for $i = 1, 2$.

Thanks to the splitting (2.13), we are able to define mild solutions to the Boltzmann equation with non-cutoff collision kernels. This is obtained by considering the Duhamel formula associated to the splitting (2.13) along the characteristics (as in the cutoff case).

Definition 2.4. Let f_0 be a measurable function, non-negative almost everywhere on $\overline{\Omega} \times \mathbb{R}^d$.

A measurable function $f = f(t, x, v)$ on $[0, T) \times \overline{\Omega} \times \mathbb{R}^d$ is a mild solution of the Boltzmann equation with non-cutoff angular collision kernel associated to the initial datum $f_0(x, v)$ if there exists $0 < \varepsilon_0 < \pi/4$ such that for all $0 < \varepsilon < \varepsilon_0$:

- (1) f is non-negative on $\overline{\Omega} \times \mathbb{R}^d$,
- (2) for every (t, x, v) in $\mathbb{R}^+ \times \Omega \times \mathbb{R}^d$:

$$s \mapsto L_\varepsilon[f(t, X_{s,t}(x, v), \cdot)](v), \quad s \mapsto Q_\varepsilon^+[f(t, X_{s,t}(x, v), \cdot), f(t, X_{s,t}(x, v), \cdot)](v)$$

$$s \mapsto S_\varepsilon[f(t, X_{s,t}(x, v), \cdot)](v), \quad s \mapsto Q_\varepsilon^1[f(t, X_{s,t}(x, v), \cdot), f(t, X_{s,t}(x, v), \cdot)](v)$$

are in $L_{loc}^1([0, T))$,

- (3) and for each $t \in [0, T)$, for all $x \in \Omega$ and $v \in \mathbb{R}^d$

$$\begin{aligned}
 (2.14) \quad f(t, x, v) &= f_0(x - vt, v) \exp \left[- \int_0^t (L_\varepsilon + S_\varepsilon) [f(s, X_{s,t}(x, v), \cdot)](v) ds \right] \\
 &\quad + \int_0^t \exp \left(- \int_s^t (L_\varepsilon + S_\varepsilon) [f(s', X_{s',s}(x, v), \cdot)](v) ds' \right) \\
 &\quad \times (Q_\varepsilon^+ + Q_\varepsilon^1) [f(s, X_{s,t}(x, v), \cdot), f(s, X_{s,t}(x, v), \cdot)](v) ds.
 \end{aligned}$$

if $t \leq t_\partial(x, v)$ or else

$$(2.15) \quad f(t, x, v) = f_\partial(t_\partial, x_\partial, v) \exp \left[- \int_{t_\partial(x, v)}^t (L_\varepsilon + S_\varepsilon) [f(s, X_{s,t}(x, v), \cdot)](v) ds \right] \\ + \int_{t_\partial(x, v)}^t \exp \left(- \int_s^t (L_\varepsilon + S_\varepsilon) [f(s', X_{s',s}(x, v), \cdot)](v) ds' \right) \\ \times (Q_\varepsilon^+ + Q_\varepsilon^1) [f(s, X_{s,t}(x, v), \cdot), f(s, X_{s,t}(x, v), \cdot)](v) ds,$$

where $t_\partial = t_\partial(x, v)$ and $x_\partial = x_\partial(x, v)$ are defined by (2.6) and (2.7) respectively.

Now we state our result.

Theorem 2.5. *Let Ω be a C^2 open bounded domain in \mathbb{R}^d with nowhere null normal vector and let f_0 be a non-negative continuous function on $\bar{\Omega} \times \mathbb{R}^d$. Let $B = \Phi b$ be a collision kernel satisfying (1.3), with Φ satisfying (1.4) or (1.5) and b satisfying (1.6) with ν in $[0, 2)$. Let $f(t, x, v)$ be a mild solution of the Boltzmann equation in $\bar{\Omega} \times \mathbb{R}^d$ on some time interval $[0, T)$, $T \in (0, +\infty]$, which satisfies*

- *f is continuous on $[0, T) \times (\bar{\Omega} \times \mathbb{R}^d - \Lambda_0)$ (Λ_0 grazing set defined by (1.7)), $f(0, x, v) = f_0(x, v)$ and $M > 0$ in (2.4);*
- *if Φ satisfies (1.4) with $\gamma \geq 0$ or if Φ satisfies (1.5), then f satisfies (2.1) and (2.3);*
- *if Φ satisfies (1.4) with $\gamma < 0$, then f satisfies (2.1), (2.2) and (2.3).*

Then for all $\tau \in (0, T)$ and for any exponent K such that

$$K > 2 \frac{\log(2 + \frac{2\nu}{2-\nu})}{\log 2},$$

there exists $C_1, C_2 > 0$, depending on τ, K, E_f, E'_f, W_f (and $L_f^{p_\gamma}$ if Φ satisfies (1.4) with $\gamma < 0$), such that

$$\forall t \in [\tau, T), \forall (x, v) \in \bar{\Omega} \times \mathbb{R}^d, \quad f(t, x, v) \geq C_1 e^{-C_2 |v|^K}.$$

Moreover, in the case $\nu = 0$, one can take $K = 2$ (Maxwellian lower bound).

As in the angular cutoff case, if we further assume that f_0 presents no vacuum area and that f has uniformly bounded local mass and entropy, our results are entirely constructive.

Corollary 2.6. *As for Corollary 2.3, if f_0 is bounded uniformly from below as well as the local mass of f , the local entropy of f is uniformly bounded from above then the conclusion of Theorem 2.5 holds true with constants being explicitly constructed in terms of $\tau, K, E_f, E'_f, W_f, H_f, L_f^{p_\gamma}$.*

3. THE CUTOFF CASE: A MAXWELLIAN LOWER BOUND

In this section we are going to prove a Maxwellian lower bound for a solution to the Boltzmann equation (1.1) in the case where the collision kernel satisfies a cutoff property.

The strategy to tackle this result follows the main idea used in [11][10][1] which relies on finding an “upheaval point” (a first minoration uniform in time and space but localised in velocity) and spreading this bound, thanks to a spreading property of the Q^+ operator, in order to include larger and larger velocities.

As described in the introduction (Section 1.2), we need a method that translates the usual spreading argument in the case of our problem and combine it with a strict positivity of the diffusion process. Roughly speaking, either the characteristic we are looking at comes from the diffusion of the boundary or the spreading will be generated on a straight line as in [10].

Thus our study will be split into three parts, which are the next three subsections

The first step (Section 3.1) is to partition the position and velocity spaces so that we have an immediate appearance of an “upheaval point” in each of those partitions.

As discussed in the introduction, the standard spreading method fails in the case of characteristics trajectories bouncing against $\partial\Omega$. We therefore study of the effects of the Maxwellian diffusion (Section 3.2).

The last one (Section 3.3) is to obtain uniform lower bounds less and less localised (and comparable to an exponential bound in the limit) in velocity. The strategy we use relies on the spreading property of the gain operator that we already used and follows the method in [10] combined with the treatment of localised bounds in [1], together with the previous focus on the Maxwellian diffusion process.

A separated part, Section 3.4, is dedicated to a constructive approach to the Maxwellian lower bound.

3.1. Initial localised lower bounds. In Section 3.1.2 we use the continuity of f together with the conservation of total mass (2.4) and the uniform boundedness of the local energy to obtain a point in the phase space where f is strictly positive. Then, thanks to the continuity of f , its Duhamel representation (2.11) – (2.10) and the spreading property of the Q^+ operator (Lemma 3.2) we extend this positivity to high velocities at that particular point (Lemma 3.3).

Finally, in Section 3.1.3, the free transport part of the solution f will imply the immediate appearance of the localised lower bounds (Proposition 3.5).

Moreover we define constants that we will use in the next two subsections in order to have a uniform lower bound.

3.1.1. Controls on the gain and the loss operators. We first introduce two lemmas, proven in [10], that control the gain and loss terms in the Grad’s splitting (2.8) we are using under the cutoff assumption. One has an L^∞ bound on the loss term (Corollary 2.2 in [10]).

Lemma 3.1. *Let g be a measurable function on \mathbb{R}^d . Then*

$$\forall v \in \mathbb{R}^d, \quad |L[g](v)| \leq C_g^L \langle v \rangle^{\gamma^+},$$

where C_g^L is defined by:

(1) If Φ satisfies (1.4) with $\gamma \geq 0$ or if Φ satisfies (1.5), then

$$C_g^L = \text{cst } n_b C_\Phi e_g.$$

(2) If Φ satisfies (1.4) with $\gamma \in (-d, 0)$, then

$$C_g^L = \text{cst } n_b C_\Phi [e_g + l_g^p], \quad p > d/(d + \gamma).$$

The spreading property of Q^+ is given by the following lemma (Lemma 2.4 in [10]), where we define

$$(3.1) \quad l_b = \inf_{\pi/4 \leq \theta \leq 3\pi/4} b(\cos \theta).$$

Lemma 3.2. *Let $B = \Phi b$ be a collision kernel satisfying (1.3), with Φ satisfying (1.4) or (1.5) and b satisfying (1.6) with $\nu \leq 0$. Then for any $\bar{v} \in \mathbb{R}^d$, $0 < r \leq R$, $\xi \in (0, 1)$, we have*

$$Q^+(\mathbf{1}_{B(\bar{v}, R)}, \mathbf{1}_{B(\bar{v}, r)}) \geq \text{cst } l_b C_\Phi r^{d-3} R^{3+\gamma} \xi^{\frac{d}{2}-1} \mathbf{1}_{B(\bar{v}, \sqrt{r^2 + R^2(1-\xi)})}.$$

As a consequence in the particular quadratic case $\delta = r = R$, we obtain

$$Q^+(\mathbf{1}_{B(\bar{v}, \delta)}, \mathbf{1}_{B(\bar{v}, \delta)}) \geq \text{cst } l_b C_\Phi \delta^{d+\gamma} \xi^{\frac{d}{2}-1} \mathbf{1}_{B(\bar{v}, \delta\sqrt{2}(1-\xi))},$$

for any $\bar{v} \in \mathbb{R}^d$ and $\xi \in (0, 1)$.

3.1.2. *First “upheaval” point.* We start by the strict positivity of our function at one point for all velocities.

Lemma 3.3. *Let f be the mild solution of the Boltzmann equation described in Theorem 2.2.*

Then there exist $\Delta > 0$, (x_1, v_1) in $\Omega \times \mathbb{R}^d$ such that for all $n \in \mathbb{N}$ there exist $r_n > 0$, depending only on n , and $t_n(t)$, $\alpha_n(t) > 0$ such that

$$\forall t \in [0, \Delta], \forall x \in B\left(x_1, \frac{\Delta}{2^n}\right), \forall v \in \mathbb{R}^d, \quad f(t, x, v) \geq \alpha_n(t) \mathbf{1}_{B(v_1, r_n)}(v),$$

with $\alpha_0 > 0$ independent of t and the induction formula

$$\alpha_{n+1}(t) = C_Q \frac{r_n^{d+\gamma}}{4^{d/2-1}} \int_{t_n(t)}^t e^{-sC_L \langle 2r_n + \|v_1\| \rangle^{\gamma^+}} \alpha_n^2(s) ds$$

where $C_Q = \text{cst } l_b C_\Phi$ is defined in Lemma 3.2 and $C_L = \text{cst } n_b C_\Phi E_f$ (or $C_L = \text{cst } n_b C_\Phi (E_f + L_f^p)$) is defined in Lemma 3.1, and

$$(3.2) \quad r_0 = \Delta, \quad r_{n+1} = \frac{3\sqrt{2}}{4} r_n,$$

$$(3.3) \quad t_n(t) = \max \left\{ 0, t - \frac{\Delta}{2^{n+1} (\|v_1\| + r_n)} \right\}.$$

Remark 3.4. *It is essentially the same method used to generate “upheaval points” in [1]. The main difference being that we need to control the characteristics before they bounce against the boundary, whence the existence of the bound $t_n(t)$.*

Proof of Lemma 3.3. The proof is an induction on n and mainly follows the method in [1] Lemma 3.3.

Step 1: Initialization. We recall the assumptions that are made on the solution f ((2.4) and assumption (2.1)):

$$\forall t \in \mathbb{R}^+, \quad \int_{\Omega} \int_{\mathbb{R}^d} f(t, x, v) \, dx dv = M, \quad \sup_{(t,x) \in [0,T) \times \Omega} \int_{\mathbb{R}^d} |v|^2 f(t, x, v) \, dx dv = E_f,$$

with $M > 0$ and $E_f < \infty$.

Since Ω is bounded, and so is included in, say, $B(0, R_X)$, we also have that

$$\forall t \in \mathbb{R}^+, \quad \int_{\Omega} \int_{\mathbb{R}^d} (|x|^2 + |v|^2) f(t, x, v) \, dx dv \leq \alpha = MR_X^2 + R_X E_f < +\infty.$$

Therefore, exactly the same computations as in [1] (step 1 of the proof of Lemma 3.3) are applicable and lead to the existence of (x_1, v_1) such that $f(0, x_1, v_1) > 0$ and, by uniform continuity of f , to Lemma 3.3 in the case $n = 0$.

Step 2: Proof of the induction. We assume the conjecture is valid for n . Let x be in $B(x_1, \Delta/2^{n+1})$, v in $B(0, \|v_1\| + 2r_n)$ and t in $[0, \Delta]$.

We have straightforwardly that

$$\forall s \in [t_n(t), t], \quad \|x_1 - (x - (t - s)v)\| \leq \frac{\Delta}{2^n}.$$

Besides, we have that $B(x_1, \Delta) \subset \Omega$ and therefore the characteristic line $(X_{s,t}(x, v))_{s \in [t_n(t), t]}$ stays in Ω . This implies that $t_{\partial}(x, v) > t_n(t)$.

We thus use the fact that f is a mild solution to write $f(t, x, v)$ under its Duhamel form starting at $t_n(t)$ without contact with the boundary (2.10). The control we have on the L operator, Lemma 3.1, allows us to bound from above the second integral term (the first term is positive). Moreover, this bound on L is independent on t , x and v since it only depends on an upper bound on the energy $e_{f(t,x,\cdot)}$ (and its local L^p norm $l_{f(t,x,\cdot)}^p$) which is uniformly bounded by E_f (and by L_f^p). This yields

$$(3.4) \quad f(t, x, v) \geq \int_{t_n(t)}^t e^{-sC_L \langle \|v_1\| + 2r_n \rangle^+} Q^+ [f(s, X_{s,t}(x, v), \cdot), f(s, X_{s,t}(x, v), \cdot)](v) \, ds,$$

where $C_L = cst n_b C_{\Phi} E_f$ (or $C_L = cst n_b C_{\Phi} (E_f + L_f^p)$), see Lemma 3.1, and we used $\|v\| \leq 2r_n + \|v_1\|$.

We already saw that $(X_{s,t}(x, v))_{s \in [t_n(t), t]}$ stays in $B(x_1, \Delta/2^n)$. Therefore, by calling v_* the integration parametre in the operator Q^+ we can apply the induction property to $f(s, X_{s,t}(x, v), v_*)$ which implies, in (3.4),

$$f(t, x, v) \geq \int_{t_n(t)}^t e^{-sC_L(\|v_1\|+2r_n)^{\gamma^+}} \alpha_n^2(s) Q^+ [\mathbf{1}_{B(v_1, r_n)}, \mathbf{1}_{B(v_1, r_n)}] ds(v).$$

Applying the spreading property of Q^+ , Lemma 3.2, with $\xi = 1/4$ gives us the expected result for the step $n + 1$ since $B(v_1, r_{n+1}) \subset B(0, \|v_1\| + 2r_n)$. \square

3.1.3. Partition of the phase space and first localised lower bounds. We are now able to prove the immediate appearance of localised “upheaval points”. We emphasize here that the following proposition is proven with exactly the same arguments as in [1].

Proposition 3.5. *Let f be the mild solution of the Boltzmann equation described in Theorem 2.2 and consider x_1, v_1 constructed in Lemma 3.3.*

Then there exists $\Delta > 0$ such that for all $0 < \tau_0 \leq \Delta$, there exists $\delta_T(\tau_0), \delta_X(\tau_0), \delta_V(\tau_0), R_{\min}(\tau_0), a_0(\tau_0) > 0$ such that for all N in \mathbb{N} there exists N_X in \mathbb{N}^ and x_1, \dots, x_{N_X} in Ω and v_1, \dots, v_{N_X} in $B(0, R_{\min}(\tau_0))$ and*

- $\overline{\Omega} \subset \bigcup_{1 \leq i \leq N_X} B(x_i, \delta_X(\tau_0)/2^N);$
 - $\forall t \in [\tau_0, \delta_T(\tau_0)], \forall x \in B(x_i, \delta_X(\tau_0)), \forall v \in \mathbb{R}^d,$
- $$f(t, x, v) \geq a_0(\tau_0) \mathbf{1}_{B(v_i, \delta_V(\tau_0, N))}(v).$$

Proof of Proposition 3.5. We are going to use the free transport part of the Duhamel form of f (2.10), to create localised lower bounds out of the one around (x_1, v_1) in Lemma 3.3.

Ω is bounded so let us denote its diameter by d_Ω .

Take τ_0 in $(0, \Delta]$. Let n be large enough such that $r_n \geq 2d_\Omega/\tau_0 + \|v_1\|$, where r_n is defined by (3.2) in Lemma 3.3. It is possible since (r_n) increases to infinity. Now, define $R_{\min}(\tau_0) = 2d_\Omega/\tau_0$.

Thanks to Lemma 3.3 applied to this particular n we have that

$$(3.5) \quad \forall t \in \left[\frac{\tau_0}{2}, \Delta\right], \forall x \in B(x_1, \Delta/2^n), \quad f(t, x, v) \geq \alpha_n\left(\frac{\tau_0}{2}\right) \mathbf{1}_{B(v_1, r_n)}(v),$$

where we used the fact that $\alpha_n(t)$ is an increasing function.

Define

$$a_0(\tau_0) = \frac{1}{2} \alpha_n\left(\frac{\tau_0}{2}\right) e^{-\frac{\tau_0}{2} C_L \left(\frac{2d_\Omega}{\tau_0}\right)^{\gamma^+}}.$$

Definition of the constants. We notice that for all x in $\partial\Omega$ we have that $n(x) \cdot (x - x_1) > 0$, because Ω has nowhere null normal vector by hypothesis. But the function

$$x \longmapsto n(x) \cdot \frac{x - x_1}{\|x - x_1\|}$$

is continuous (since Ω is C^2) on the compact $\partial\Omega$ and therefore has a minimum that is attained at a certain $X(x_1)$ on $\partial\Omega$.

Hence,

$$(3.6) \quad \forall x \in \partial\Omega, \quad n(x) \cdot \frac{x - x_1}{\|x - x_1\|} \geq n(X(x_1)) \cdot \frac{X(x_1) - x_1}{\|X(x_1) - x_1\|} = 2\lambda(x_1) > 0.$$

To shorten following notations, we define on $\overline{\Omega} \times (\mathbb{R}^d - \{0\})$ the function

$$(3.7) \quad \Phi(x, v) = n \left(x + t \left(x, \frac{v}{\|v\|} \right) \frac{v}{\|v\|} \right),$$

where we defined $t(x, v) = \min\{t \geq 0 : x + tv \in \partial\Omega\}$, the first time of contact against the boundary of the forward characteristic $(x + sv)_{s \geq 0}$ defined for $v \neq 0$ and continuous on $\overline{\Omega} \times (\mathbb{R}^d - \{0\})$ (see [1] Lemma 5.2 for instance).

We denote d_1 to be half of the distance from x_1 to $\partial\Omega$. We define two sets included in $[0, \Delta] \times \overline{\Omega} \times \mathbb{R}^d$:

$$\Lambda^{(1)} = [0, \Delta] \times B(x_1, d_1) \times \mathbb{R}^d$$

and

$$\Lambda^{(2)} = \left\{ (t, x, v) \notin \Lambda^{(1)}, \quad \|v\| \geq \frac{d_1}{\tau_0} \quad \text{and} \quad \Phi(x, v) \cdot \frac{v}{\|v\|} \geq \lambda(x_1) \right\}$$

By continuity of $t(x, v)$ and of n (on $\partial\Omega$), we have that

$$\Lambda = \Lambda^{(1)} \cap \Lambda^{(2)}$$

is compact and does not intersect the grazing set $[0, \Delta] \times \Lambda_0$ defined by (1.7). Therefore, f is continuous in Λ and thus is uniformly continuous on Λ . Hence, there exist $\delta'_T(\tau_0), \delta'_X(\tau_0), \delta'_V(\tau_0) > 0$ such that

$$(3.8) \quad \forall (t, x, v), (t', x', v') \in \Lambda, \quad |t - t'| \leq \delta'_T(\tau_0), \|x - x'\| \leq \delta'_X(\tau_0), \|v - v'\| \leq \delta'_V(\tau_0), \\ |f(t, x, v) - f(t', x', v')| \leq a_0(\tau_0).$$

The map Φ (defined by (3.7)) is uniformly continuous on the compact $[0, \Delta] \times \overline{\Omega} \times \mathbb{S}^{d-1}$ and therefore there exist $\delta''_T(\tau_0), \delta''_X(\tau_0), \delta''_V(\tau_0) > 0$ such that

$$(3.9) \quad \forall (t, x, v), (t', x', v') \in \Lambda^{(2)}, \quad |t - t'| \leq \delta''_T(\tau_0), \|x - x'\| \leq \delta''_X(\tau_0), \|v - v'\| \leq \delta''_V(\tau_0), \\ |\Phi(x, v) - \Phi(x', v')| \leq \frac{\lambda(x_1)}{2}.$$

We conclude our definitions by taking

$$\begin{aligned} \delta_T(\tau_0) &= \min(\Delta, \tau_0 + \delta'_T(\tau_0), \tau_0 + \delta''_T(\tau_0)), \\ \delta_X(\tau_0) &= \min\left(\frac{\Delta}{2^n}, \delta'_X(\tau_0), \delta''_X(\tau_0), d_1/2\right), \\ \delta_V(\tau_0) &= \min\left(r_n, \delta'_V(\tau_0), \frac{d_1}{2\tau_0}\delta''_V(\tau_0), \frac{\lambda(x_1)}{2}\right). \end{aligned}$$

Proof of the lower bounds. We take $N \in \mathbb{N}$ and notice that $\overline{\Omega}$ is compact and therefore there exists x_1, \dots, x_{N_X} in Ω such that $\overline{\Omega} \subset \bigcup_{1 \leq i \leq N_X} B(x_i, \delta_X(\tau_0)/2^N)$.

Moreover, we construct them such that x_1 is the one defined in Lemma 3.3 and we then take v_1 to be the one defined in Lemma 3.3. We define

$$\forall i \in \{2, \dots, N_X\}, \quad v_i = \frac{2}{\tau_0}(x_i - x_1).$$

Because Ω is convex we have that

$$\begin{aligned} X_{\tau_0/2, \tau_0}(x_i, v_i) &= x_1, \\ V_{\tau_0/2, \tau_0}(x_i, v_i) &= v_i. \end{aligned}$$

The latter equalities imply that there is no contact with $\partial\Omega$ between times $\tau_0/2$ and τ_0 when starting from x_1 to go to x_i with velocity v_i . Using the fact that f is a mild solution of the Boltzmann equation, we write it under its Duhamel form without contact (2.10), but starting at $\tau_0/2$. We drop the last term which is positive. As in the proof of Lemma 3.3 we can control the L operator appearing in the first term in the right-hand side of (2.10) (corresponding to the free transport).

$$\begin{aligned} f(\tau_0, x_i, v_i) &\geq f\left(\frac{\tau_0}{2}, x_1, v_i\right) e^{-\frac{\tau_0}{2} C_L \langle \frac{2}{\tau_0}(x_i - x_1) \rangle^{\gamma^+}} \\ &\geq \alpha_n \left(\frac{\tau_0}{2}\right) e^{-\frac{\tau_0}{2} C_L \langle \frac{2d\Omega}{\tau_0} \rangle^{\gamma^+}} \mathbf{1}_{B(v_1, r_n)}(v_i) \\ &\geq 2a_0(\tau_0) \mathbf{1}_{B(v_1, r_n)}(v_i), \end{aligned}$$

where we used (3.5) for the second inequality. We see here that v_i belongs to $B(0, R_{\min}(\tau_0))$ and that $B(0, R_{\min}(\tau_0)) \subset B(v_1, r_n)$ and therefore

$$(3.10) \quad f(\tau_0, x_i, v_i) \geq 2a_0(\tau_0).$$

We first notice that (τ_0, x_i, v_i) belongs to Λ since either x_i belongs to $B(x_1, d_1)$ or $\|x_1 - x_i\| \geq d_1$ but by definition of v_i and $\lambda(x_1)$ (see (3.6)),

$$n \left(x_i + t \left(x_i, \frac{v_i}{\|v_i\|} \right) \frac{v_i}{\|v_i\|} \right) \cdot \frac{v_i}{\|v_i\|} \geq 2\lambda(x_1)$$

and

$$\|v_i\| = \frac{2}{\tau_0} \|x_i - x_1\| \geq \frac{2}{\tau_0} d_1.$$

We take t in $[\tau_0, \delta_T(\tau_0)]$, x in $B(x_i, \delta_X(\tau_0))$ and v in $B(v_i, \delta_V(\tau_0))$ and we will prove that (t, x, v) also belongs to Λ .

If x_i belongs to $B(x_1, d_1/2)$ then since $\delta_X(\tau_0) \leq d_1/2$,

$$\|x - x_1\| \leq \frac{d_1}{2} + \|x - x_i\| \leq d_1$$

and (t, x, v) thus belongs to $\Lambda^{(1)} \subset \Lambda$.

In the other case where $\|x_1 - x_i\| \geq d_1/2$ we first have that

$$\|v_i\| = \frac{2}{\tau_0} \|x_i - x_1\| \geq \frac{d_1}{\tau_0}.$$

And also

$$\left\| \frac{v}{\|v\|} - \frac{v_i}{\|v_i\|} \right\| \leq \frac{2}{\|v_i\|} \|v - v_i\| = \frac{\tau_0}{\|x_i - x_1\|} \delta_V(\tau_0) \leq \frac{2\tau_0}{d_1} \delta_V(\tau_0) \leq \delta_V''(\tau_0).$$

The latter inequality combined with (3.9) and that $|t - \tau_0| \leq \delta_T''(\tau_0)$ and $\|x - x_i\| \leq \delta_X''(\tau_0)$ yields

$$|\Phi(x, v) - \Phi(x_i, v_i)| \leq \frac{\lambda(x_1)}{2},$$

which in turn implies

$$\begin{aligned} \Phi(x, v) \cdot \frac{v}{\|v\|} &\geq \Phi(x_i, v_i) \cdot \frac{v_i}{\|v_i\|} + \Phi(n, v) \cdot (v - v_i) + (\Phi(x, v) - \Phi(x_i, v_i)) \cdot v_i \\ &\geq 2\lambda(x_1) - \|v - v_i\| - |\Phi(x, v) - \Phi(x_i, v_i)| \\ &\geq \lambda(x_1), \end{aligned}$$

so that (t, x, v) belongs to $\Lambda^{(2)}$.

We can now conclude the proof.

We proved that (τ_0, x_i, v_i) belongs to Λ and that for all t in $[\tau_0, \delta_T(\tau_0)]$, x in $B(x_i, \delta_X(\tau_0))$ and v in $B(v_i, \delta_V(\tau_0))$, (t, x, v) belongs to Λ . By definition of the constants, $(t - \tau_0, x - x_i, v - v_i)$ satisfies the inequality of the uniform continuity of f on Λ (3.8). Combining this inequality with (3.10), the lower bound at (τ_0, x_i, v_i) , we have that

$$f(t, x, v) \geq a_0(\tau_0).$$

□

Remark 3.6. *In order to lighten our presentation and because τ_0 can be chosen as small as one wants, we will only study the case of solutions to the Boltzmann equation which satisfies Proposition 3.5 at $\tau_0 = 0$. Then we will immediately create the exponential lower bound after at τ_1 for all $\tau_1 > 0$. Then we apply the latter result to $F(t, x, v) = f(t + \tau_0, x, v)$ to obtain the exponential lower bound for f at time $\tau_0 + \tau_1$ which can be chosen as small as one wants.*

3.2. Global strict positivity of the Maxwellian diffusion. In this subsection, we focus on the positivity of the Maxwellian diffusion boundary condition. More precisely, we prove that the boundary of the domain Ω diffuses in all directions a minimal strictly positive quantity.

Proposition 3.7. *Let f be the mild solution of the Boltzmann equation described in Theorem 2.2 and consider $\Delta > 0$ constructed in Proposition 3.5. Then for all τ_0 in $(0, \Delta]$, there exists $b_\partial(\tau_0) > 0$ such that*

$$\forall t \in [\tau_0, \Delta], \forall x_\partial \in \partial\Omega, \quad \left[\int_{v_* \cdot n(x_\partial) > 0} f(t, x_\partial, v_*) (v_* \cdot n(x_\partial)) \, dv_* \right] > b_\partial(\tau_0).$$

Proof of Proposition 3.7. Let τ_0 be in $(0, \Delta]$, take t in $[\tau_0, \Delta]$ and x_∂ on $\partial\Omega$.

We consider (x_1, v_1) constructed in Lemma 3.3 and will use the same notations as in the proof of Proposition 3.5.

In the spirit of the proof of Proposition 3.5 we define

$$v_\partial = \frac{2}{t}(x_\partial - x_1),$$

which gives, because $\partial\Omega$ is convex, that $(X_{t/2,s})_{t/2 \leq s \leq t}$ is a straight line not intersecting $\partial\Omega$ apart at time $s = t$. We can thus write f under its Duhamel form without contact (2.10) starting at $t/2$. We keep only the first term on the right hand-side and control the operator L by Lemma 3.1.

$$\begin{aligned} f(t, x_\partial, v_\partial) &\geq f\left(\frac{t}{2}, x_1, v_\partial\right) e^{-\frac{t}{2}C_L\langle\frac{2}{t}(x_\partial - x_1)\rangle^{\gamma^+}} \\ &\geq \alpha_n\left(\frac{t}{2}\right) e^{-\frac{t}{2}C_L\langle\frac{2d\Omega}{t}\rangle^{\gamma^+}} \mathbf{1}_{B(v_1, r_n)}(v_\partial) \\ &\geq \alpha_n\left(\frac{\tau_0}{2}\right) e^{-\frac{\Delta}{2}C_L\langle\frac{2d\Omega}{\tau_0}\rangle^{\gamma^+}} \mathbf{1}_{B(v_1, r_n)}(v_\partial), \end{aligned}$$

where we used (3.5) for the second inequality since $t/2$ belongs to $[\tau_0/2, \Delta]$. Note that we choose n exactly as in the proof of Proposition 3.5 and, for the same reasons, v_∂ thus belongs to $B(v_1, r_n)$ which implies

$$f(t, x_\partial, v_\partial) \geq \alpha_n\left(\frac{\tau_0}{2}\right) e^{-\frac{\Delta}{2}C_L\langle\frac{2d\Omega}{\tau_0}\rangle^{\gamma^+}}.$$

Here again, the continuity of f away from the grazing set implies the existence of $\delta(\tau_0)$ independent of t , x and v_∂ such that

$$(3.11) \quad \forall v_* \in B(v_\partial, \delta(\tau_0)), \quad f(t, x_\partial, v_*) \geq \frac{1}{2}\alpha_n\left(\frac{\tau_0}{2}\right) e^{-\frac{\Delta}{2}C_L\langle\frac{2d\Omega}{\tau_0}\rangle^{\gamma^+}} = A(\tau_0) > 0.$$

We now deal with the scalar product $v_* \cdot n(x_\partial)$ appearing in the Maxwellian diffusion.

We notice that for all x in $\partial\Omega$ we have that $n(x) \cdot (x - x_1) > 0$, because Ω has nowhere null normal vector by hypothesis. But the function $x \mapsto n(x) \cdot (x - x_1)$ is continuous (since Ω is C^2) on the compact $\partial\Omega$ and therefore has a minimum that is attained at a certain $X(x_1)$ on $\partial\Omega$.

Hence,

$$\forall x \in \partial\Omega, \quad (x - x_1) \cdot n(x) \geq n(X(x_1)) \cdot (X(x_1) - x_1) = 2B(x_1) > 0.$$

We define $\delta'(x_1) = B(x_1)$ and a mere Cauchy-Schwarz inequality implies that

$$\forall x \in \partial\Omega, \forall v_* \in B((x - x_1), \delta'(x_1)), \quad v_* \cdot n(x) \geq B(x_1) > 0,$$

which in turns implies

$$(3.12) \quad \forall t \in [\tau_0, \Delta], \forall x \in \partial\Omega, \forall v_* \in B\left(\frac{2}{t}(x - x_1), \frac{2}{\Delta}\delta'(\tau_0)\right), \quad v_* \cdot n(x) \geq \frac{2}{\Delta}B(x_1).$$

To conclude we combine (3.11) and (3.12) at point x_∂ inside the integrale to get, with $\delta''(\tau_0) = \min(\delta(\tau_0), 2\delta'(x_1)/\Delta)$,

$$\begin{aligned}
\int_{v_* \cdot n(x_\partial) > 0} f(t, x_\partial, v_*) (v_* \cdot n(x_\partial)) \, dv_* &\geq \int_{v_* \in B(v_\partial, \delta''(\tau_0))} f(t, x_\partial, v_*) (v_* \cdot n(x_\partial)) \, dv_* \\
&\geq \frac{2}{\Delta} A(\tau_0) B(x_1) |B(v_\partial, \delta''(\tau_0))| \\
&= \frac{2}{\Delta} A(\tau_0) B(x_1) |B(0, \delta''(\tau_0))|.
\end{aligned}$$

This yields the expected result with

$$(3.13) \quad b_\partial(\tau_0) = 2A(\tau_0)B(x_1) |B(0, \delta''(\tau_0))| / \Delta.$$

□

Remark 3.8. For now on, τ_0 used in Proposition 3.7 will be the same as the one used in Proposition 3.5. Moreover, as already mentioned and explained in Remark 3.6, we will consider in the sequel that $\tau_0 = 0$.

3.3. Spreading of the initial localised bounds and global Maxwellian lower bound. In Section 3.1 we proved the immediate appearance of initial lower bounds that are localised in space and in velocity. The present subsection aims at increasing these lower bounds in a way that larger and larger velocities are taken into account, and compare it to an exponential bound in the limit.

3.3.1. Spreading of the initial “upheaval points”. First, we pick N in \mathbb{N}^* , construct δ_V , R_{min} and cover $\overline{\Omega}$ with $\bigcup_{1 \leq i \leq N_X} B(x_i, \delta_X/2^N)$ as in Proposition 3.5 where we dropped the dependencies in τ_0 and N .

Then for any sequence (ξ_n) in $(0, 1)$ and for all $\tau > 0$ we define three sequences in \mathbb{R}^+ by induction. First,

$$(3.14) \quad \begin{cases} r_0 &= \delta_V \\ r_{n+1} &= \sqrt{2} (1 - \xi_n) r_n. \end{cases}$$

Second, with the notation $\tilde{r}_n = R_{min} + r_n$,

$$(3.15) \quad \begin{cases} b_0(\tau) &= b_\partial \\ b_{n+1}(\tau) &= b_\partial e^{-C_L \tau \langle \tilde{r}_{n+1} \rangle^{\gamma^+}} \frac{e^{-\frac{(\tilde{r}_{n+1})^2}{2T_\partial}}}{(2\pi)^{\frac{d-1}{2}} T_\partial^{\frac{d+1}{2}}}, \end{cases}$$

where b_∂ defined in Proposition 3.7 and C_L in Lemma 3.3. And finally, with C_Q being defined in Lemma 3.3,

$$(3.16) \quad \begin{cases} a_0(\tau) &= a_0 \\ a_{n+1}(\tau) &= \min(a_n(\tau), b_n(\tau))^2 C_Q r_n^{d+\gamma} \xi_{n+1}^{d/2-1} \frac{\tau}{2^{n+2} \tilde{r}_{n+1}} e^{-C_L \frac{\tau \langle \tilde{r}_{n+1} \rangle^{\gamma^+}}{2^{n+2} \tilde{r}_{n+1}}}. \end{cases}$$

We express the spreading of the lower bound in the following proposition.

Proposition 3.9. *Let f be the mild solution of the Boltzmann equation described in Theorem 2.2 and suppose that f satisfies Proposition 3.5 with $\tau_0 = 0$. Consider $0 < \tau \leq \delta_T$ and N in \mathbb{N} . Let $(x_i)_{i \in \{1, \dots, N_X\}}$ and $(v_i)_{i \in \{1, \dots, N_X\}}$ be given as in Proposition 3.5 with $\tau_0 = 0$. Then for all n in $\{0, \dots, N\}$ we have that the following holds:*

$$\forall t \in \left[\tau - \frac{\tau}{2^{n+1}\tilde{r}_n}, \tau \right], \forall x \in B\left(x_i, \frac{\delta_X}{2^n}\right), \quad f(t, x, v) \geq \min(a_n(\tau), b_n(\tau)) \mathbf{1}_{B(v_i, r_n)}(v),$$

where (r_n) , (a_n) and (b_n) are defined by (3.14)-(3.16)-(3.15).

Proof of Proposition 3.9. We are interested in immediate appearance of a lower bound so we can always choose $\delta_t \leq \delta_X$ and also $R_{min} \geq 1$.

This Proposition will be proved by induction on n , the initialisation is simply Proposition 3.5 so we consider the case where the proposition is true for $n < N$.

Take $t \in [\tau - \tau/(2^{n+2}\tilde{r}_{n+1}), \tau]$, $x \in B(x_i, \delta_X/2^{n+1})$ and $v \in B(v_i, 2r_{n+1})$.

There are two possible cases depending on $t_\partial(x, v) \leq t$ or $t_\partial(x, v) > t$. We remind here that t_∂ (see (2.6)) is the first time of contact of the backward trajectory starting at x with velocity v .

1st case: no contact against the boundary from 0 to t . We can therefore use the Duhamel formula without contact (2.10) and bound it by the second term on the right-hand side (every term being positive). Then Lemma 3.1 on the linear operator L and the fact that

$$\|v\| \leq \|v_i\| + r_{n+1} \leq R_{min} + r_{n+1} = \tilde{r}_{n+1},$$

gives the following inequality

$$(3.17) \quad f(t, x, v) \geq \int_{\tau - \frac{\tau}{2^{n+1}\tilde{r}_{n+1}}}^{\tau - \frac{\tau}{2^{n+2}\tilde{r}_{n+1}}} e^{-C_L(t-s)\langle \tilde{r}_{n+1} \rangle^{\gamma^+}} Q^+ [f(s, X_{s,t}(x, v), \cdot), f(s, X_{s,t}(x, v), \cdot)] ds.$$

We now see that for all s in the interval considered in the integral above

$$\|x_i - X_{s,t}(x, v)\| \leq \|x_i - x\| + |t - s| \|v\| \leq \frac{\delta_X}{2^{n+1}} + \frac{\tau}{2^{n+1}} \leq \frac{\delta_X}{2^n},$$

where we used that $\delta_T \leq \delta_X$. We can therefore apply the induction hypothesis to $f(t, X_{s,t}(x, v), v_*)$ into (3.17), where we denoted by v_* the integration parameter in Q^+ . This yields

$$f(t, x, v) \geq a_n(\tau)^2 e^{-C_L \frac{\tau \langle \tilde{r}_{n+1} \rangle^{\gamma^+}}{2^{n+2}\tilde{r}_{n+1}}} \left(\int_{\tau - \frac{\tau}{2^{n+1}\tilde{r}_{n+1}}}^{\tau - \frac{\tau}{2^{n+2}\tilde{r}_{n+1}}} Q^+ [\mathbf{1}_{B(v_i, r_n)}, \mathbf{1}_{B(v_i, r_n)}] ds \right) (v).$$

Applying the spreading property of Q^+ , Lemma 3.2, with $\xi = \xi_{n+1}$ gives us the expected result for the step $n + 1$ with the lower bound $a_{n+1}(\tau)$.

2nd case: there is at least one contact before time t . In that case we have that $t_\partial(x, v) \leq t$ and we can use the Duhamel formula with contact (2.11) for f . Both terms on the right-hand side are positive so we can lower bound $f(t, x, v)$ by the first one. Denoting $x_\partial = x_\partial(x, v)$ (see Definition (2.7)), this yields

$$(3.18) \quad \begin{aligned} f(t, x, v) &\geq e^{-C_L t \langle \tilde{r}_{n+1} \rangle^{\gamma^+}} f_\partial(t_\partial(x, v), x_\partial(x, v), v) \\ &\geq \frac{e^{-\frac{(\tilde{r}_{n+1})^2}{2T_\partial}}}{(2\pi)^{\frac{d-1}{2}} T_\partial^{\frac{d+1}{2}}} e^{-C_L \tau \langle \tilde{r}_{n+1} \rangle^{\gamma^+}} \left[\int_{v_* \cdot n(x_\partial) > 0} f(t, x_\partial, v_*) (v_* \cdot n(x_\partial)) dv_* \right]. \end{aligned}$$

where we used the definition of f_∂ (1.2).

Thanks to the previous subsection (Proposition 3.7), we obtain straightforwardly the expected result for step $n + 1$ with the lower bound $b_{n+1}(\tau)$. \square

3.3.2. A Maxwellian lower bound: proof of Theorem 2.2. In this subsection we prove Theorem 2.2.

We take f being the mild solution described in Theorem 2.2 and we suppose, thanks to Remarks 3.6 and 3.8, that f satisfies Propositions 3.5 and 3.7 with $\tau_0 = 0$.

We fix $\tau > 0$ and we keep the notations defined in Section 3.3.1 for the sequences (r_n) , (a_n) and (b_n) (see (3.14)-(3.16)-(3.15), with (ξ_n) to be defined later).

In Proposition 3.9, we showed that we can spread the localised lower bound with larger and larger velocities taken into account, i.e. by taking $\xi_n = 1/4$ the sequence r_n is strictly increasing to infinity. We can consider $r_0 > 0$ and find an n_0 in \mathbb{N} such that $B(0, R_{min})$

$$\forall v \in B(0, R_{min}), \quad B(0, r_0) \subset B(v, r_{n_0}).$$

By setting N to be this specific n_0 and applying Proposition 3.9 with this N we obtain a uniform lower bound:

$$\forall t \in \left[\tau - \frac{\tau}{2^{n_0+1} \tilde{r}_{n_0}}, \tau \right], \quad \forall x \in \overline{\Omega}, \quad f(t, x, v) \geq \min(a_{n_0}(\tau), b_{n_0}(\tau)) \mathbf{1}_{B(0, r_{n_0})}(v).$$

This bound is uniform in x and the same arguments as in the proof of Proposition 3.9 allows us to spread it in the same manner in an even easier way since it is a global lower bound. Therefore, without loss of generality, we can assume $n_0 = 0$ and that the following holds,

$$\forall n \in \mathbb{N}, \quad \forall t \in \left[\tau - \frac{\tau}{2^{n+1} \tilde{r}_n}, \tau \right], \quad \forall x \in \overline{\Omega}, \quad f(t, x, v) \geq \min(a_n(\tau), b_n(\tau)) \mathbf{1}_{B(0, r_n)}(v),$$

with (r_n) , (a_n) and (b_n) satisfying the same inductive properties (3.14)-(3.16)-(3.15) (with $r_0 = r_{n_0}$, $a_0 = a_{n_0}$ and $b_0 = b_{n_0}$), with (ξ_n) to be chosen later.

The proof of Theorem 2.2 is then done in two steps. The first one is to establish the Maxwellian lower bound at time τ using a slightly modified version of the argument in [10] Lemma 3.3. The second is to prove that the latter bound holds for all $t > \tau$.

1st step: a Maxwellian lower bound at time τ . A natural choice for (ξ_n) is a geometric sequence $\xi_n = \xi^n$ for a given ξ in $(0, 1)$. With such a choice we have that

$$(3.19) \quad r_n \leq r_0 2^{\frac{n}{2}}$$

and

$$(3.20) \quad r_n = 2^{\frac{n}{2}} r_0 \prod_{k=1}^n (1 - \xi^k) \geq c_r 2^{\frac{n}{2}},$$

with $c_r > 0$ depending only on r_0 and ξ .

It follows that f satisfies the following property

$$(3.21) \quad \forall n \in \mathbb{N}, \forall x \in \overline{\Omega}, \forall v \in B(0, c_r 2^{\frac{n}{2}}), \quad f(\tau, x, v) \geq c_n,$$

with $c_n = \min(a_n, b_n)$.

It has been proven in [10] Lemma 3.3 that for a function satisfying the property (3.21) with $c_n \geq \alpha^{2^n}$, for some $\alpha > 0$, there exist ρ and θ strictly positive explicit constants such that

$$\forall x \in \overline{\Omega}, \forall v \in \mathbb{R}^d, \quad f(\tau, x, v) \geq \frac{\rho}{(2\pi\theta)^{d/2}} e^{-\frac{|v|^2}{2\theta}}.$$

It thus only remains to show that there exists α_1 and α_2 strictly positive such that $b_n \geq \alpha_1^{2^n}$ and $a_n \geq \alpha_2^{2^n}$.

The case of (b_n) is quite straightforward from (3.19) and $0 \leq \gamma^+ \leq 1$. Indeed, there exist an explicit constants C_1 and $C_2 > 0$, independent of n , such that for all $n \geq 1$

$$\begin{aligned} b_n &= b_\partial e^{-C_L \tau \langle \tilde{r}_n \rangle^{\gamma^+}} \frac{e^{-\frac{(\tilde{r}_n)^2}{2T_\partial}}}{(2\pi)^{\frac{d-1}{2}} T_\partial^{\frac{d+1}{2}}} \geq \frac{b_\partial}{2T_\partial (2\pi)^{\frac{d-1}{2}}} e^{-C_1 (R_{\min} + r_n)^2} \\ &\geq C_2 e^{-2C_1 r_0^2 2^n}. \end{aligned}$$

Therefore if $C_2 \geq 1$ we define $\alpha_1 = \min(b_\partial, e^{-2C_1 r_0^2})$ or else we define $\alpha_1 = \min(b_\partial, C_2 e^{-2C_1 r_0^2})$ and it yields $b_n \geq \alpha_1^{2^n}$ for all $n \geq 0$.

We recall the inductive definition of (a_n) for $n \geq 0$, with $\xi_n = \xi^n$,

$$a_{n+1} = \min(a_n, b_n)^2 C_Q r_n^{d+\gamma} \xi^{(n+1)(d/2-1)} \frac{\tau}{2^{n+2} \tilde{r}_{n+1}} e^{-C_L \frac{\tau \langle \tilde{r}_{n+1} \rangle^{\gamma^+}}{2^{n+2} \tilde{r}_{n+1}}}.$$

First, using (3.19) we have that for all $n \geq 0$,

$$\begin{aligned} -C_L \frac{\tau \langle \tilde{r}_{n+1} \rangle^{\gamma^+}}{2^{n+2} \tilde{r}_{n+1}} &\leq \frac{C_L \tau R_{\min}^{\gamma^+}}{2R_{\min}} + \frac{C_L \tau r_{n+1}^{\gamma^+}}{2^{n+2} R_{\min}} \\ &\leq \frac{C_L \tau R_{\min}^{\gamma^+}}{2R_{\min}} + \frac{C_L \tau r_0^{\gamma^+} 2^{\frac{(n+1)\gamma^+}{2}}}{2^{n+2} R_{\min}}, \end{aligned}$$

which is bounded from above for all n since $0 \leq \gamma^+ \leq 1$. Therefore, if we denote by C_3 any explicit non-negative constant independent of n , we have for all $n \geq 0$

$$a_{n+1} \geq C_3 \frac{r_n^{d+\gamma} \xi^{(n+1)(d/2-1)}}{2^{n+2} (R_{\min} + r_{n+1})} \min(a_n, b_n)^2.$$

Thus using (3.19) and (3.20) to bound r_n we have

$$\begin{aligned}
a_{n+1} &\geq C_3 \frac{2^{\frac{n(d+\gamma)}{2}} \xi^{(n+1)(d/2-1)}}{2^{n+2}(R_{\min} + c_r 2^{\frac{n+1}{2}})} \min(a_n, b_n)^2 \\
&\geq C_3 \frac{2^{\frac{n(d+\gamma)}{2}} \xi^{n(d/2-1)}}{2^{\frac{3n+5}{2}}} \min(a_n, b_n)^2 \\
&\geq C_3 \left(\frac{2^{\frac{(d+\gamma)}{2}} \xi^{(d/2-1)}}{2^{\frac{3}{2}}} \right)^n \min(a_n, b_n)^2.
\end{aligned}$$

We define

$$\lambda = \min(1, C_3) \frac{2^{\frac{(d+\gamma)}{2}} \xi^{(d/2-1)}}{2^{\frac{3}{2}}},$$

which leads to,

$$(3.22) \quad \forall n \geq 1, \quad a_{n+1} \geq \lambda^n \min(a_n, b_n)^2.$$

We could always have chosen b_0 and then b_1 respectively smaller than a_0 and a_1 (by always bounding from below by the minimum) and we assume that it is so. We can therefore define

$$\forall n \geq 1, \quad k_n = \min \{0 \leq k \leq n-1 : a_{n-k} \geq b_{n-k}\}.$$

Notice that $n - k_n \geq 1$, hence (3.22) can be iterated k_n times which yields

$$\begin{aligned}
\forall n \geq 1, \quad a_{n+1} &\geq \lambda^{n+2(n-1)+\dots+2^{k_n}(n-k_n)} \min(a_{n-k_n}, b_{n-k_n})^{2^{k_n+1}} \\
&\geq \lambda^{2^{k_n+1}(n-k+1)-(n+2)} \left(\alpha_1^{2^{n-k_n}} \right)^{2^{k_n+1}}.
\end{aligned}$$

Thus, if $\lambda \geq 1$ we can choose $\alpha_2 = \alpha_1$ and else we have $2^{k_n+1}(n-k+1)-(n+2) \leq 2^{n+1}$ and we can choose $\alpha_2 = \lambda \alpha_1$. In any case, α_2 does not depend on n and we have $a_n \geq \alpha_2^{2^n}$.

We therefore proved the existence of $\alpha > 0$ such that for all n in \mathbb{N} , $\min(a_n, b_n) \geq \alpha^{2^n}$ which implies that there exist ρ and θ strictly positive explicit constants such that

$$(3.23) \quad \forall x \in \overline{\Omega}, \forall v \in \mathbb{R}^d, \quad f(\tau, x, v) \geq \frac{\rho}{(2\pi\theta)^{d/2}} e^{-\frac{|v|^2}{2\theta}}.$$

2nd step: a Maxwellian lower bound for all $T > t \geq \tau$. To complete the proof of Theorem 2.2, it remains to prove that (3.23) actually holds for all $t \geq \tau$. All the results and constants we obtained so far do not depend on an explicit form of f_0 but just on uniform bounds and continuity that are satisfied at all times, positions and velocities (by assumption). Therefore, we can do the same arguments starting at any time and not $t = 0$. So if we take $\tau > 0$ and consider $\tau \leq t < T$ we just have to make the proof start at $t - \tau$ to obtain Theorem 2.2.

3.4. A constructive approach to the initial lower bound and positivity of the diffusion. In previous subsections, we can see that explicit and constructive constants are obtained from given initial lower bounds and uniform positivity of the Maxwellian diffusion. Therefore, a constructive approach to the latter two will lead to completely explicit constants in the Maxwellian lower bound, depending only on *a priori* bounds on the solution and the geometry of the domain.

Localised “upheaval points”. A few more assumptions on f_0 and f suffice to obtain a completely constructive approach for the “upheaval points”. This method is based on a property of the iterated Q^+ operator discovered by Pulvirenty and Wennberg [11] and reformulated by Mouhot ([10] Lemma 2.3) as follows.

Lemma 3.10. *Let $B = \Phi b$ be a collision kernel satisfying (1.3), with Φ satisfying (1.4) or (1.5) and b satisfying (1.6) with $\nu \leq 0$. Let $g(v)$ be a nonnegative function on \mathbb{R}^d with bounded energy e_g and entropy h_g and a mass ρ_g such that $0 < \rho_g < +\infty$. Then there exist $R_0, r_0, \eta_0 > 0$ and $\bar{v} \in B(0, R_0)$ such that*

$$Q^+ (Q^+ (g \mathbf{1}_{B(0, R_0)}, g \mathbf{1}_{B(0, R_0)}), g \mathbf{1}_{B(0, R_0)}) \geq \eta_0 \mathbf{1}_{B(\bar{v}, r_0)},$$

with R_0, r_0, η_0 being constructive in terms on ρ_g, e_g and h_g .

We now suppose that $0 < \rho_{f_0} < +\infty, h_{f_0} < +\infty$ and that

$$\forall (x, v) \in \Omega \times \mathbb{R}^d, \quad f_0(x, v) \geq \varphi(v) > 0$$

and we consider R_0, r_0, η_0 and \bar{v} from Lemma 3.10 associated to the function φ .

We consider x_1 is in Ω and we denote $d_1 = d(x_1, \partial\Omega)$ the distance between x_1 and $\partial\Omega$. Define $\Delta_1 = \min(1, d_1/3R_0)$.

Take $0 < \tau_0 \leq \Delta_1$ and v in $B(0, R_0)$.

We have, by construction that

$$(3.24) \quad \forall t \in [0, \Delta_1], \quad \|x_1 - (x_1 - vt)\| \leq \frac{d_1}{3},$$

which means that $t_\partial(x, v) > \Delta_1$. By the Duhamel form without contact (2.10) of f and Lemma 3.1 we have for all t in $[\tau_0, \Delta_1]$,

$$(3.25) \quad f(t, x_1, v) \geq f_0(x, v) e^{-tC_L \langle v \rangle^{\gamma^+}} \geq \varphi(v) e^{-tC_L \langle R_0 \rangle^{\gamma^+}}$$

and

$$\begin{aligned} f(t, x_1, v) &\geq \int_0^t e^{-(t-s)C_L \langle v \rangle^{\gamma^+}} Q^+ [f(s, x_1 - (t-s)v, \cdot), f(s, x_1 - (t-s)v, \cdot)](v) ds \\ (3.26) \quad &\geq e^{-(\Delta_1)C_L \langle R_0 \rangle^{\gamma^+}} \\ &\quad \int_0^t Q^+ [f(s, x_1 - (t-s)v, \cdot) \mathbf{1}_{B(0, r_0)}, f(s, x_1 - (t-s)v, \cdot) \mathbf{1}_{B(0, r_0)}](v) ds. \end{aligned}$$

Now we notice that (3.24) implies the following.

$$\forall s \in [0, t], \quad d(x_1 - (t-s)v, \partial\Omega) \geq 2d_1/3.$$

Hence, if we call v_* the integral variable in Q^+ , we have that for all v_* in $B(0, R_0)$ and all s in $[0, t]$, $t_\partial(x - (t-s)v, v_*) > t$ (same arguments as for (3.24)). The function $f(s, x_1 - (t-s)v, v_*)$ thus satisfies (3.25) and (3.26) as well.

We can do this iteration one more time since $R_0\Delta_1 \leq d_1/3$ and this yields

$$f(t, x_1, v) \geq \tau_0^2 e^{-(3\Delta_1)C_L \langle R_0 \rangle^{\gamma^+}} \int_0^t Q^+ \left(Q^+ \left(\varphi \mathbf{1}_{B(0, R_0)}, \varphi \mathbf{1}_{B(0, R_0)} \right), \varphi \mathbf{1}_{B(0, R_0)} \right) (v) \, ds.$$

Applying Lemma 3.10 and remembering that $\Delta_1 \leq 1$ we obtain that

$$(3.27) \quad \forall t \in [\tau_0, \Delta_1], \forall v \in B(0, R_0), \quad f(t, x_1, v) \geq \tau_0^3 e^{-3C_L \langle R_0 \rangle^{\gamma^+}} \eta_0 \mathbf{1}_{B(\bar{v}, r_0)}.$$

f is uniformly continuous on the compact $[0, T/2] \times \bar{\Omega} \times B(0, R_0)$ so there exists $\delta_T, \delta_X, \delta_V > 0$ such that

$$\forall |t - t'| \leq \delta_T, \forall \|x - x'\| \leq \delta_X, \forall \|v - v'\| \leq \delta_V,$$

$$(3.28) \quad |f(t, x, v) - f(t', x', v')| \leq a_0(\tau_0),$$

where we defined $2a_0(\tau_0) = \tau_0^3 e^{-3C_L \langle R_0 \rangle^{\gamma^+}} \eta_0$.

From (3.27) and (3.28), we find

$$(3.29) \quad \forall t \in [\tau_0, \Delta_1], \forall x \in B(x_1, \delta_X), \forall v \in B(0, R_0), \quad f(t, x, v) \geq \tau_0^3 e^{-3C_L \langle R_0 \rangle^{\gamma^+}} \eta_0 \mathbf{1}_{B(\bar{v}, r_0)}.$$

To conclude we construct x_2, \dots, x_{N_X} such that $\bar{\Omega} \subset \bigcup_{1 \leq i \leq N_X} B(x_i, \delta_X)$. We can use exactly the same arguments as for x_1 on each x_i on the time interval $[\tau_0, \Delta_i]$ and we reach the conclusion (3.29) on each $B(x_i, \delta_X)$. This gives, with $\Delta = \min(\Delta_i)$,

$$(3.30) \quad \forall t \in [\tau_0, \Delta], \forall x \in \bar{\Omega}, \forall v \in B(0, R_0), \quad f(t, x, v) \geq a_0(\tau_0) \mathbf{1}_{B(\bar{v}, r_0)}.$$

Remark 3.11. *We emphasize here that even though we used compactness arguments, they appeared to be solely a technical artifice. The constants $a_0(\tau_0)$, \bar{v} , R_0 and r_0 are entirely explicit and depends on a priori bounds on f . The only point of concern would be that Δ is not constructive since it depends on the covering. However, in previous sections, only the constant bounding the diffusive process (Proposition 3.7 contains a dependency in Δ (see (3.11) and (3.13)) but depend on an upper bound on Δ which is less than 1.*

Starting from this explicit bound we can use the proofs made in the previous subsections, that are constructive, to therefore have a completely constructive proof as long as the bound on the diffusion (Proposition 3.7) can be obtained without compactness arguments.

Constructive approach of positivity of diffusion. A quick look at the proof of Proposition 3.7 shows that we only need to construct $\delta(\tau_0)$ in (3.11) and $\delta'(x_1)$ in (3.12) explicitly.

The first equality (3.11) is obtained constructively combining the arguments to obtain it pointwise (see proof of Proposition 3.7) together with the method of the proof of Lemma 3.3.

Indeed, take x_∂ on $\partial\Omega$ and fix an x_0 in ω . We can grow the initial lower bound (3.30) at x_1 for t in $[\tau_0, \Delta]$ such that it includes $B(0, 2d_\Omega/\tau_0 + \|\bar{v}\|)$ (as in Lemma 3.3). Then, as in the beginning of the proof of Proposition 3.5 we obtain that,

$$(3.31) \quad \forall t \in [\tau_0, \Delta], \quad f\left(t, x_\partial, 2\frac{x_\partial - x_0}{\tau_0}\right) \geq A > 0.$$

We can now do that for all v in

$$B\left(2\frac{x_\partial - x_0}{\tau_0}, \min\left(r_0; 2\frac{d(x_0, \partial\Omega)}{\tau_0}\right)\right)$$

by just defining, for each v , $x(v)$ the point in Ω such that

$$v = 2\frac{x_\partial - x(v)}{\tau_0}.$$

Note that this point is always well defined since Ω is convex and

$$\|x_0 - x(v)\| \leq d(x_0, \partial\Omega).$$

For any given $x(v)$ we apply the same argument as for x_0 so that the lower bound includes v . Therefore, there is an infimum constant A_{min} satisfying,

$$\forall t \in [\tau_0, \Delta_{min}], \forall v \in B\left(2\frac{x_\partial - x_0}{\tau_0}, \min\left(r_0; 2\frac{d(x_0, \partial\Omega)}{\tau_0}\right)\right), \quad f(t, x_\partial, v) \geq A_{min} > 0,$$

which is a constructive version of (3.11). We emphasize here that A_{min} exists and is indeed independent on v since it depends on the number of iteration of Lemma 3.3 (itself determined by the norm of v which is bounded) and the initial lower bound at $x(v)$ which is uniform in space by (3.30).

The second inequality (3.12) is purely geometric as long as we fix a x_1 satisfying the initial lower bound, which is the case with x_0 used above. We therefore obtained an entirely constructive method for the positivity of the diffusion process.

4. THE NON-CUTOFF CASE: AN EXPONENTIAL LOWER BOUND

In this section we prove the immediate appearance of an exponential lower bound for solutions to the Boltzmann equation (1.1) in the case of a collision kernel satisfying the non-cutoff property.

The definition of being a mild solution in the case of a non-cutoff collision kernel, Definition 2.4 and equation (2.13), shows that we are in fact dealing with a cutoff kernel to which we add a non locally integrable remainder. As we shall see in Section 4.1, S_ε enjoys the same kind of L^∞ control than the operator L whilst Q_ε^1 , the non-cutoff part of the gain operator, has an L^∞ -norm that decreases to zero as ε goes to zero.

The strategy used is therefore utterly identical to the cutoff case: creation of localised ‘‘upheaval points’’ and spreading of these initial lower bounds up to an exponential lower bound. The difference will be that, at each step n of the spreading

process we will choose an ε_n and a ξ_n such that the perturbation added by the non-cutoff part $- \|Q_{\varepsilon_n}^1\|_{L_v^\infty}$ still preserves a uniform positivity in larger and larger balls in velocity.

Note that the uniform positivity of the Maxwellian diffusion still holds in the non-cutoff case since it only comes from an initial positivity and the geometry of the domain.

4.1. Controls on the operators S_ε and Q_ε^1 . We gather here two lemmas, proven in [10], which we shall use in this section. They control the L^∞ -norm of the linear operator S_ε and of the bilinear operator Q_ε^1 . We first give a property satisfied by the linear operator S , (2.13), which is Corollary 2.2 in [10], where we define

$$(4.1) \quad m_b = \int_{\mathbb{S}^{d-1}} b(\cos \theta) (1 - \cos \theta) d\sigma = |\mathbb{S}^{d-2}| \int_0^\pi b(\cos \theta) (1 - \cos \theta) \sin^{d-2} \theta d\theta.$$

Lemma 4.1. *Let g be a measurable function on \mathbb{R}^d . Then*

$$\forall v \in \mathbb{R}^d, \quad |S[g](v)| \leq C_g^S \langle v \rangle^{\gamma^+},$$

where C_g^S is defined by:

- (1) *If Φ satisfies (1.4) with $\gamma \geq 0$ or if Φ satisfies (1.5), then*

$$C_g^S = \text{cst } m_b C_\Phi e_g.$$

- (2) *If Φ satisfies (1.4) with $\gamma \in (-d, 0)$, then*

$$C_g^S = \text{cst } m_b C_\Phi [e_g + l_g^p], \quad p > d/(d + \gamma).$$

We will compare the lower bound created by the cutoff part of our kernel to the remaining part Q_ε^1 . To do so we need to control its L^∞ -norm. This is achieved thanks to Lemma 2.5 in [10], which we recall here.

Lemma 4.2. *Let $B = \Phi b$ be a collision kernel satisfying (1.3), with Φ satisfying (1.4) or (1.5) and b satisfying (1.6) with $\nu \in [0, 2)$. Let f, g be measurable functions on \mathbb{R}^d .*

Then

- (1) *If Φ satisfies (1.4) with $2 + \gamma \geq 0$ or if Φ satisfies (1.5), then*

$$\forall v \in \mathbb{R}^d, \quad |Q_b^1(g, f)(v)| \leq \text{cst } m_b C_\Phi \|g\|_{L_{\tilde{\gamma}}^1} \|f\|_{W^{2,\infty}} \langle v \rangle^{\tilde{\gamma}}.$$

- (2) *If Φ satisfies (1.4) with $2 + \gamma < 0$, then*

$$\forall v \in \mathbb{R}^d, \quad |Q_b^1(g, f)(v)| \leq \text{cst } m_b C_\Phi \left[\|g\|_{L_{\tilde{\gamma}}^1} + \|g\|_{L^p} \right] \|f\|_{W^{2,\infty}} \langle v \rangle^{\tilde{\gamma}}$$

with $p > d/(d + \gamma + 2)$.

4.2. Proof of Theorem 2.5. As explained at the beginning of this section, the main idea is to compare the loss due to the non-cutoff part of the operator Q_ε^1 with the spreading properties of the cutoff operator Q_ε^+ . More precisely, due to Lemmas 3.1, 4.1 and 4.2 we find that for all $0 < \varepsilon < \varepsilon_0$,

$$(4.2) \quad L_\varepsilon[f] + S_\varepsilon[f] \leq C_f (n_{b_\varepsilon^{CO}} + m_{b_\varepsilon^{NCO}}) \langle v \rangle^{\gamma^+}$$

and

$$(4.3) \quad Q_\varepsilon^+(f, f) + Q_\varepsilon^1(f, f) \geq Q_\varepsilon^+(f, f) - |Q_\varepsilon^1(f, f)| \geq Q_\varepsilon^+(f, f) - C_f m_{b_\varepsilon^{NCO}} \langle v \rangle^{(2+\gamma)^+},$$

where $C_f > 0$ is a constant depending on E_f , E'_f , W_f (and $L_f^{p_\gamma}$ if Φ satisfies (1.4) with $\gamma < 0$).

Moreover, by definitions (3.1), (2.9) and (4.1), the following behaviours happen:

$$l_{b_\varepsilon^{CO}} \geq l_b$$

and

$$(4.4) \quad n_{b_\varepsilon^{CO}} \underset{\varepsilon \rightarrow 0}{\sim} \frac{b_0}{\nu} \varepsilon^{-\nu}, \quad m_{b_\varepsilon^{NCO}} \underset{\varepsilon \rightarrow 0}{\sim} \frac{b_0}{2-\nu} \varepsilon^{2-\nu}$$

if ν belongs to $(0, 2)$ and

$$(4.5) \quad n_{b_\varepsilon^{CO}} \underset{\varepsilon \rightarrow 0}{\sim} b_0 |\log \varepsilon|, \quad m_{b_\varepsilon^{NCO}} \underset{\varepsilon \rightarrow 0}{\sim} \frac{b_0}{2} \varepsilon^2$$

for $\nu = 0$. This shows that the contribution of Q_ε^1 decreases with ε so this operator should not affect the spreading method whereas the contribution of S_ε increases, which is why we lose the Maxwellian lower bound to get a faster exponential one.

We just briefly describe the changes to make into the proof of the cutoff case to obtain Theorem 2.5.

Localised “upheaval points”. The creation of localised initial lower bounds (Proposition 3.5 in the cutoff case) depends on the Boltzmann operator for two different reasons:

- the creation of a first lower bound in a neighborhood of a point (x_1, v_1) in the phase space (Lemma 3.3)
- the creation of localised lower bounds in $\overline{\Omega}$ via the free transport part.

Since $L_\varepsilon + S_\varepsilon$ satisfies the same bounds as L in the cutoff case, the second step can be made identically as in the proof of Proposition 3.5. It remains to prove the creation of an initial bound in a neighborhood of (x_1, v_1) .

We use the same definition of Δ , x_1 , v_1 , $\alpha_n(t)$ and $(r_n)_{n \in \mathbb{N}}$ as in Lemma 3.3 apart from

$$t_n(t) = \max \left\{ \tau_0, t - \frac{\Delta}{2^{n+1} (\|v_1\| + r_n)} \right\}.$$

Note that the step $n = 0$ holds here since it only depends on the continuity of the solution f and f_0 .

The key difference will be that the equivalent statement of Lemma 3.3 can only be done on time intervals of the form $[\tau_0, \Delta]$ for any $\tau_0 > 0$. Indeed, take $\tau_0 > 0$ and therefore

$$\forall n \in \mathbb{N}, \forall t \in [\tau_0, \Delta], \quad \alpha_n(t) \geq \alpha_n(\tau_0).$$

Exactly the same arguments as in the inductive step $n+1$ in the proof of Lemma 3.3 we reach (3.4) with our new operators (with a cutoff function depending on ε_{n+1})

$$f(t, x, v) \geq \int_{t_n(t)}^t e^{-sC_f^{\varepsilon_{n+1}}(R)} \left(\alpha_n(\tau_0)^2 Q_{\varepsilon_{n+1}}^+ [\mathbf{1}_{B(\bar{v}, r_n)}, \mathbf{1}_{B(\bar{v}, r_n)}] - C_f m_{b_{\varepsilon_{n+1}}^{NCO}} \langle R \rangle^{(2+\gamma)^+} \right) (v) ds,$$

using the shorthand notations $C_f^\varepsilon(R) = C_f(n_{b_\varepsilon^{CO}} + m_{b_\varepsilon^{NCO}}) \langle R \rangle^{\gamma^+}$ and $R = \|v_1\| + 2r_n$. Due to the spreading property of $Q_{\varepsilon_{n+1}}^+$ (see Lemma 3.2) with $\xi = 1/4$ we reach

$$(4.6) \quad f(t, x, v) \geq \int_{t_n(t)}^t e^{-C_f^{\varepsilon_{n+1}}(R)} \left(\alpha_n^2(\tau_0) \text{cst } l_{b_{\varepsilon_{n+1}}^{CO}} c_\Phi r_n^{d+\gamma} \xi^{\frac{d}{2}-1} \mathbf{1}_{B(\bar{v}, r_n \sqrt{2}(1-\xi))} - C_f m_{b_{\varepsilon_{n+1}}^{NCO}} \langle R \rangle^{(2+\gamma)^+} \right) (v) ds,$$

Thus, at each step of the induction we just have to choose ε_{n+1} small enough such that

$$(4.7) \quad C_f m_{b_{\varepsilon_{n+1}}^{NCO}} \langle R \rangle^{(2+\gamma)^+} \leq \frac{1}{2} \alpha_n^2(\tau_0) \text{cst } l_{b_{\varepsilon_{n+1}}^{CO}} c_\Phi r_n^{d+\gamma} \xi^{\frac{d}{2}-1}.$$

This proves the following Lemma.

Lemma 4.3. *Let f be the mild solution of the Boltzmann equation described in Theorem 2.5.*

Then there exist $\Delta > 0$, (x_1, v_1) in $\Omega \times \mathbb{R}^d$ such that for all $n \in \mathbb{N}$ there exist $r_n > 0$, depending only on n , such that for all τ_0 in $(0, \Delta]$ there exists and $\alpha_n(\tau_0) > 0$ such that for

$$\forall t \in [\tau_0, \Delta], \forall x \in B\left(x_1, \frac{\Delta}{2^n}\right), \forall v \in \mathbb{R}^d, \quad f(t, x, v) \geq \alpha_n(\tau_0) \mathbf{1}_{B(v_1, r_n)}(v).$$

Exponential lower bound. As explained before, the strict positivity of the diffusion still holds in our case since we proved the initial lower bound in Lemma 4.3. It therefore remains to show that we can indeed spread the “upheaval points”. This is achieved by adapting the arguments of the cutoff case together with careful choices of ε_{n+1} and ξ_{n+1} at each step of the induction. This has been done in [10] and in [1] and we refer to these works for deeper details.

Basically, we start by spreading the initial “upheaval points” (obtained from Lemma 4.3 with the same method as Proposition 3.5) by induction. At each step of the induction we use the spreading property of the $Q_{\varepsilon_n}^+$ operator between $t_n^{(2)}$ and $t_n^{(1)}$ (see (3.17)) and we fix ε_n small enough to obtain a strictly positive lower bound (see (4.7)).

There is, however, a subtlety in the non-cutoff case that we have to deal with. Indeed, at each step of the induction we choose an ε_n of decreasing magnitude, but at the same time in each step the action of the operator $-(L_{\varepsilon_n} + S_{\varepsilon_n})$ behaves like (see (4.6))

$$\exp \left[-C_f \left(m_{b_{\varepsilon_n}^{NCO}} + n_{b_{\varepsilon_n}^{CO}} \right) (t_n^{(1)} - t_n^{(2)}) \langle v \rangle^{\gamma^+} \right].$$

By (4.4) – (4.5), as ε_n tends to 0 we have that $n_{b_{\varepsilon_n}^{CO}}$ goes to $+\infty$ and so the action of $-(Q_\varepsilon^- + Q_\varepsilon^2)$ seems to decrease the lower bound to 0 exponentially fast. The idea to overcome this difficulty is to find a time interval $t_n^{(1)} - t_n^{(2)} = \Delta_n$, at each step to be sufficiently small to counterbalance the effect of $n_{b_{\varepsilon_n}^{CO}}$.

More precisely, taking

$$t_n^{(1)} = \left(\sum_{k=0}^{n+1} \Delta_k \right) \tau, \quad t_n^{(2)} = \left(\sum_{k=0}^n \Delta_k \right) \tau \quad \text{with} \quad \sum_{k=0}^{\infty} \Delta_k = 1,$$

fixing ε_n by (4.7) and choosing carefully ξ_n (exactly as in [10][1]) we reach the desired Theorem 2.5.

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